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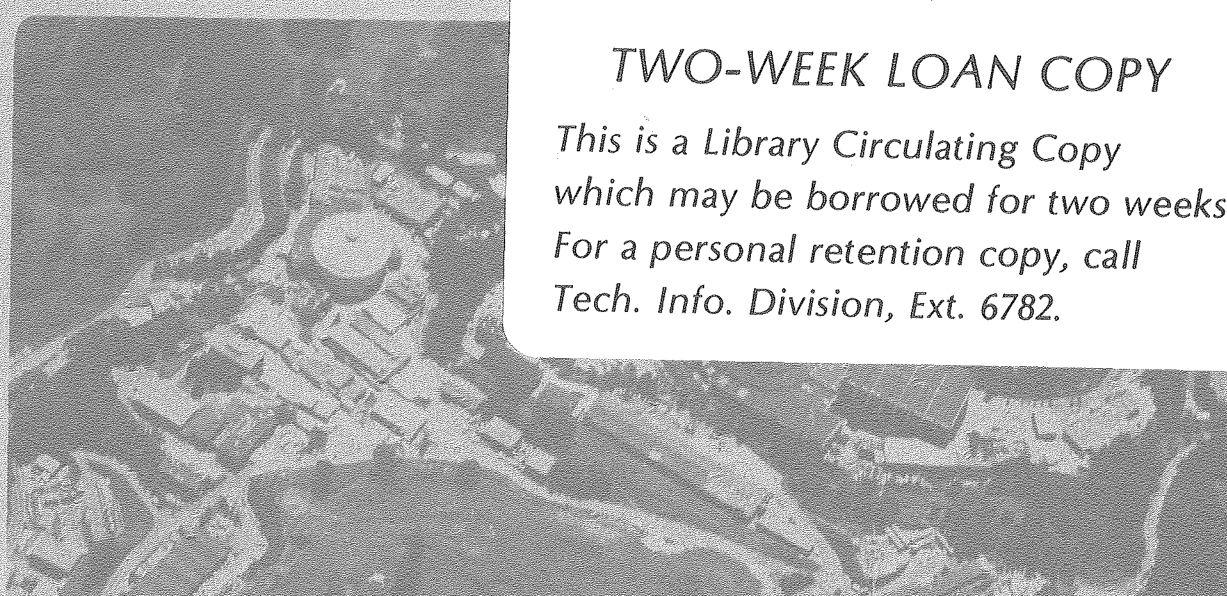



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## DISCLAIMER

# THE $CP^{N-1}$ MODEL WITH UNCONSTRAINED VARIABLES[†]

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ABSTRACT

The $CP^{N-1}$ model is investigated in terms of unconstrained variables in both the Lagrangian and Hamiltonian formulations. The presence of a gauge ambiguity is intimately related with confinement and the appearence of a dynamically generated vector field. Various aspects of the spectrum of states in the large $N$ limit are investigated.

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## INTRODUCTION

The study of models in which particles appear in non-linear representations has played an important role in several areas in particle physics. In recent years[1,2] the emphasis has shifted from studying the non-linear aspects to uncovering the linear properties of naively nonlinear models. In particular attempts have been made to uncover a linear version of QCD [3] (at least in the limit of large $N$). In this paper we wish to investigate similar questions in such a model, the $CP^{n-1}$ model. There has recently been much interest in this model[4,5,6]. The model is asymptotically free, possesses instantons for any $N$, is expandable in $1/N$, and exhibits a non-trivial confinement mechanism. All this has led to speculation that the model may be a testing ground for some ideas in QCD. The analysis of the model as carried out previously is in terms of variables with a constraint.[4] The Lagrangian

$$\mathcal{L} = (\partial_\mu \bar{z})(\partial_\mu z) + f/2N\,(\bar{z}\overleftrightarrow{\partial}_\mu z)^2$$

$$\text{with}\quad \bar{z}_i z_i = N/2f$$

$z$ is a column vector with $N$ entries and $f$ is a constant. The generating functional was examined by introducing dummy fields $\alpha$ (to impose the constrant) and

$$A_\mu = if\,\frac{\bar{z}\overleftrightarrow{\partial}_\mu z}{\sqrt{N}}$$

(to render the Lagrangian quadratic in $z$). A $1/N$ expansion then reveals that to leading order in $1/N$, all $N$ $z_i$ fields get a non-zero mass $m$, and to next order a long range force appears (the $A_\mu$ field acquires dynamics) which leads to confinement of the $z_i$ particles into $SU(N)$ multiplets.

Motivated by the non-uniformity of the $1/N$ limit, the physics being determined by the next to leading term, and the non-trivial way in which the $A_\mu$ field is introduced, we determined to investigate the model in terms of unconstrained variables . By fixing a gauge and transforming to a non-linear representation of $CP^{N-1}$, we are able to write an equivalent Lagangian in terms of unconstrained fields. The resulting Lagrangian has a gauge ambiguity not dissimilar to the Gribov ambiguity in QCD.[7] The Lagrangian is then analyzed in a $1/N$ expansion and the appearance of the gauge field $A_\mu$ is intimately related to this gauge ambiguity. Section 2 is concerned with this Lagrangian formulation.

In Section 3, we investigate the Hamiltonian derived from our Lagrangian. In the $O(N)/O(N-1)$ non-linear sigma model, which is contained within our Hamiltonian by dropping some terms, we are able to obtain the vacuum state and the restoration of the symmetry. In the $CP^{N-1}$ case the Hamiltonian has a singularity associated with the gauge ambiguity which plays a crucial role and provides an immediate signal for confinement.

In Section 4 we analyze the spectrum of the model in the $1/N$ limit, pointing out that the lowest lying states form an adjoint representation of $SU(N)$. We comment on the connection between this spectrum and the one computed in the limit of strong coupling limit of the lattice version of the model as formulated by Stone.[8]



## 2. LAGRANGIAN FORMULATION

The Lagrangian of the $CP^{N-1}$ model as written in terms of constrained fields[4] is

$$\mathcal{L} = (\partial_\mu \bar{z})(\partial_\mu z) + \lambda/4\ (\bar{z}\overleftrightarrow{\partial}_\mu z)^2 \tag{2.1}$$

$$\text{with } \bar{z}z = 4/\lambda$$

$z_i$ is an $N$ component column vector, in $\bar{z}z$ a sum over the index i is implied. We have defined $\lambda = 2f/N$. In order to re-write the Lagrangian in terms of unconstrained fields it is nessesary to go to a non-linear representation of the coset space

$$CP^{N-1} \simeq SU(N)/S(U(1)\times U(N-1))$$

The $z_i$'s transform linearly with respect to $SU(N)$. A non-linear transformation on the $z_i$ variables of the type

$$z_i = f(\bar{\phi}\phi)\phi_i$$

where $f$ is any function of $\bar{\phi}\phi$ for which $f(1) = 1$ will preserve the S-matrix of the $z_i$ fields. The simplest non-linear transformation which leads also to unconstrained variables $\phi_i$ is $\phi_i = z_i/z_N$ for $i = 1, \ldots\ldots N-1$. Unfortunately this transformation is singular when $z_N = 0$, a set of configurations for which the model reduces to the $CP^{N-2}$ model. Symbolically we may write $CP^{N-1} = CP^{N-2} \cup C^{N-1}$, where $C^{N-1}$ corresponds to the unconstrained variables and the $CP^{N-2}$ corresponds to the singular point $z_N = 0$. We shall use a non-linear representation which naively has no such problem. The Lagrangian 2.1 is invariant under local gauge transformations $z \to e^{i\lambda(x)}z$; we will use this invariance to fix $z_N$ to be real for all $x$ (by obvious analogy we refer to this as the unitary gauge).* We next use Schwinger's[9] choice and define unconstrained $\phi_i$ for $i = 1, \ldots N-1$ by

$$z_i = \phi_i/(1 + \frac{\lambda\bar{\phi}\phi}{4})$$

The real $z_N$ field is determined through the constraint $\bar{z}_i z_i = 4/\lambda$ $i = 1, \ldots N$. This transformation was used by Bardeen[10] et. al. for the $O(N)/O(N-1)$ non-linear sigma model. With this choice of gauge

$$\mathcal{L} = \frac{(\partial_\mu \bar{\phi})(\partial_\mu \phi)}{(1 + \frac{\lambda}{4}\bar{\phi}\phi)^2} + \frac{\lambda}{4}\frac{(\bar{\phi}\overleftrightarrow{\partial_\mu}\phi)^2}{(1 + \frac{\lambda}{4}\bar{\phi}\phi^2)^4} \qquad (2.2)$$

$\phi$ is now unconstrained and usual field theory methods can be applied to it. In particular a Hamiltonian can be derived (see next section) for the system. The Lagrangian 2.2 as expressed in terms of unconstrained fields $\phi$ bears a strong resemblence to the original Lagrangian (Eq. 2.1), as expressed in terms of the $N$ complex constrained $z_i$ fields. In fact in the subset of the constrained $\phi$ fields obeying the constraint

$$\bar{\phi}\phi = 4/\lambda$$

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* We are restricting ourselves to the $\Theta = 0$ sector by this singular gauge transformation.



the Lagrangian still possesses a local gauge invariance. Note that this ambiguity, which we refer to as a Gribov ambiguity, exists only for large field configurations. The infra-red finite perturbative calculations, using massless $z_i$ fields but group singlets, (e.g. calculation of the $\beta$ function) are not affected by this ambiguity. Since

$$z_N = \sqrt{\frac{1}{\lambda}}\,(1 - \frac{\lambda}{4}\bar{\phi}\phi)/(1 + \frac{\lambda}{4}\bar{\phi}\phi)$$

it is clear that the reason for this residual gauge invariance is that when $z_N = 0$, the specification of the phase is meaningless and does not fix the gauge at all. This difficulty is analagous to the problem at $z_N = 0$ in the simple projective transformation discussed earlier. While the source of this ambiguity is clear its physical implications depend on the dynamics of the system. A priori, the complete classification of the "ambigious" configurations leads to a set of zero measure. Hovever in the large $N$ limit, we find that $\bar{\phi}\phi$ has a vacuum expectation value:

$$\langle 0|\bar{\phi}\phi|0\rangle = 4/\lambda$$

The off-shell pole at $s = m^2$ corresponds to the appearance of another particle degenerate with the $\phi$'s and a full restoration of the SU(N) symmetry. The last terms, the cut of $T_{22}$, may be viewed as a short range potential be between the $\phi$'s, we will return to it later.

Now consider the contribution of the $i = j = 3$ part to $\Gamma^{(4)}$. Firstly $\Sigma_{33}$ which corresponds to the graph shown in Figure must be evaluated,

$$B_{\mu\nu}(q^2) = \frac{\lambda N}{2} \int \frac{(2k + q)_\mu \ (2k - q)_\nu}{(k^2 - m^2)((k - q)^2 - m^2)} \ \frac{d^d k}{(2\pi)^d} \ .$$

The integral diverges, however we can evaluate it by means of the renormalization prescription Eq. 2.3.

$$B_{\mu\nu} = g_{\mu\nu} - \frac{\lambda N}{2} \Gamma_{\mu\nu}(q^2)$$

where

$$\Gamma_{\mu\nu}(q^2) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \left[(- q^2 + 4m^2)A(q^2) \ - \frac{1}{\pi}\right]$$

$$A(q^2) = \frac{1}{2\pi} \ \frac{1}{(q^2(q^2 - 4m^2))^{\frac{1}{2}}} \log \left[\frac{(q^2 - 4m^2)^{\frac{1}{2}} - \sqrt{q^2}}{(q^2 - 4m^2)^{\frac{1}{2}} + \sqrt{q^2}}\right]$$

The bubble sum is given by $-i\lambda(p_1 + p_2)_\mu T_{\mu\nu}(p_3 + p_4)_\nu$ with

$$T_{\mu\nu} = B_{\mu\nu} + B_{\mu\alpha} T_{\alpha\nu}$$



i.e.

$$\begin{aligned} T_{\mu\nu} &= (g_{\mu\alpha} - B_{\mu\alpha})^{-1} B_{\alpha\nu} \\ &= \frac{2}{\lambda N} \ \frac{\left(g_{\mu\alpha} - \frac{q_\mu q_\alpha}{q^2}\right)^{-1}}{((4m^2 - q^2)A(p) - 1/\pi)} \left(g_{\alpha\nu} - \frac{\lambda N}{2} \Gamma_{\alpha\nu}(q^2)\right) \end{aligned} \quad (2.5)$$

It is clear that we have encounted a non-invertable term. This term is characteristic of a vector field with a gauge invariance. It will be necessary to further fix the gauge in order to proceed. This problem can be resolved because the vacuum has chosen the ambiguous field configurations. We have the freedom to further fix the gauge by choosing a Lorentz gauge for $\bar{\phi}\overleftrightarrow{\partial}_\mu\phi$ when $\bar{\phi}\phi = 4/\lambda$ which will enable us to invert equation 2.5 i.e.

$$\partial_\mu(\bar{\phi}\overleftrightarrow{\partial}_\mu\phi) \ = \ 0$$

then

$$T_{\mu\nu}(q^2) = \frac{2}{\lambda N} \frac{g_{\mu\nu}}{((4m^2 - q^2)A(q^2) - 1/\pi)} \ . \quad (2.6)$$

The full vertex is then obtained by inserting $T_{\mu\nu}$ into both $t$ and $s$ channels

$$i\lambda \left[ (p_1 + p_2)_\mu T_{\mu\nu}(s)(p_3 + p_4)_\nu + (p_1 - p_3)_\mu T_{\mu\nu}(t)(p_2 - p_4)_\nu \right] \quad (2.7)$$

and adding this term to Eq. 2.4

We can now compare with the work of D'Adda et. al.[4]. They obtained an effective Lagrangian in which the interaction between the z particles (which now have a mass as a result of the symmetry restoration) interact via a short range scalar interaction (called $\alpha$) and a long range gauge field (called $A_\mu$). The short range piece $\alpha$ is already present in the $O(N)/O(N-1)$ case, it corresponds to the last term in Eq. 2.4 and is the cut piece of the $\partial_\mu\bar{\phi}\partial_\mu\phi$ propogator. The gauge field produces an interaction identical to that of Eq. 2.7. That is for small $q^2$ all constants in the numerator of Eq. 2.6 cancel and $T_{\mu\nu} \sim g_{\mu\nu}/q^2$. A weak long range force appears which radically alters the spectrum of the system. We note that we have added a Lorentz condition on the configurations determining the photon propogator, this is justified for $\phi$ such that $\bar{\phi}\phi = 4/\lambda$. The information we used was only that the expectation value of $\bar{\phi}\phi$ is $4/\lambda$ in the $1/N$ vacuum. However for $N \to \infty$ the theory is essentially a free field theory of $\bar{\phi}\phi$ bound states and all its disconnected Green functions factorize. This means that a certain set of classical configurations determines all the bound state Green functions in this limit, and in particular the set $\bar{\phi}\phi = 4/\lambda$ dominates the functional integral.

In our formulation the two fields $\alpha$ and $A_\mu$ appear naturally and the fact that the gauge field acquires real dynamics is exemplified by the fact that it is necessary to further fix a Lorentz type gauge. Having obtained the complete long range force in a Lagrangian formulation, we turn to the Hamiltonian formulation in which the caution which needs to be exercised in a $1/N$ expansion is more explicit.



## 3. HAMILTONIAN FORMULATION

A Hamiltonian formulation of the $CP^{N-1}$ model is made possible by the use of unconstrained variables. The Hamiltonian will be set up in the unitary gauge; attempts to work in an axial gauge will be discussed briefly later. The canonical momenta $\pi_i$ derived from the Lagrangian 2.2 are

$$\pi_i = \frac{\delta \mathcal{L}}{\delta\left(\frac{\partial\bar{\phi}_i}{\partial t}\right)} = \frac{1}{D^2}\left[\dot{\phi}_i - \frac{\lambda}{2D^2}(\bar{\phi}_j\dot{\phi}_j - \dot{\bar{\phi}}_j\phi_j)\phi_i\right]$$

with

$$D = \left(1 + \frac{\lambda\bar{\phi}\phi}{4}\right)$$

and the dot denotes differentation with respect to time. We are able to construct the Hamiltonian

$$H = D^2(\bar{\pi}\pi - \frac{\lambda}{4}(\bar{\phi}\pi - \bar{\pi}\phi)\frac{1}{\left(1 - \frac{\lambda\bar{\phi}\phi}{4}\right)}(\bar{\phi}\pi - \bar{\pi}\phi))$$

$$+ \frac{1}{D^2}\left[(\partial_x\bar{\phi})(\partial_x\phi) + \frac{\lambda}{4D^2}(\bar{\phi}(\partial_x\phi) - (\partial_x\bar{\phi})\phi)^2\right]$$

Several comments should be made concerning this Hamiltonian. The ordering of the operators is not defined. Fortunately problems associated with operator ordering will not appear in leading order in $1/N$. There is a pole in the Hamiltonian when $\bar{\phi}\phi = 4/\lambda$, the field configurations for which a residual gauge invariance exists. The residue at this pole which is $-\lambda(\bar{\phi}\pi - \bar{\pi}\phi)$

is proportional to the charge operator. This is reminiscent of Mandelstam's[2] and Gribov's[7] treatement of the QCD Hamiltonian in the Coulomb gauge. A term of the form $\rho \frac{1}{\Delta} \rho$ appears there.* Care should be taken to avoid infinite energy states (states in which $\frac{1}{\Delta}$ has non-zero support). The appearence of a term containing on the one hand a pole and on the other hand a factor of 1/N is a warning signal in the way of a 1/N expansion. The physical consequences of this structure will be determined by the specific dynamics of the system. Despite this ambiguity, we wish to emphasize that our Hamiltonian will have no problems if it is used for perturbation theory about the vacuum $\phi^2 = 0$. We thus proceed to study the Hamiltonian in the naive 1/N limit. In that case, assuming that the operator $(\bar{\phi}\pi - \bar{\pi}\phi)$ has zero vacuum expectation value (no breakdown of charge conservation), the model then reduces to the 1/N limit of the O(N)/O(N-1) model. We will attempt to construct the vacuum state of that system by means of a Hartree approximation. The O(N)/O(N-1) case is obtained by dropping the last two terms in the Hamiltonian (equivalent to the naive 1/N limit), leaving

$$H_{O(N)} = D^2 \bar{\pi}\pi + \frac{1}{D^2} (\partial_x \bar{\phi})(\partial_x \phi).$$

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* The operator $\Delta$ is defined in Ref. 7 as $\Delta \equiv \Box \frac{1}{\partial^2} \Box$ .



The vacuum energy of this Hamiltonian can be obtained from the trial decomposition

$$\phi_i(x,t=0) = \int \frac{dk}{2\pi\sqrt{2\omega_k}} \left(a_k^i e^{ikx} + b_k^{\dagger i} e^{-ikx}\right)$$

$$\bar{\phi}_i(x,t=0) = \int \frac{dk}{2\pi\sqrt{2\omega_k}} \left(b_k^i e^{ikx} + a_k^{\dagger i} e^{-ikx}\right)$$

$$\bar{\pi}_i(x,t=0) = -i\int \sqrt{\frac{\omega_k}{2}} \frac{dk}{2\pi} \left(b_k^i e^{ikx} - a_k^{\dagger i} e^{-ikx}\right)$$

$$\pi_i(x,t=0) = -i\int \sqrt{\frac{\omega_k}{2}} \frac{dk}{2\pi} \left(a_k^i e^{ikx} - b_k^{\dagger i} e^{-ikx}\right)$$

with

$$\left[a_k^{i\dagger}, a_q^j\right] = \delta_{ij}\delta(k-q)$$

$$\left[b_k^{i\dagger}, b_q^j\right] = \delta_{ij}\delta(k-q)$$

and all other commutators zero. The vacuum energy $\langle 0|H|0\rangle = E_{vac}$ is a functional of $\omega_k$ given by

$$E_{vac} \underset{N\to\infty}{=} \left(\frac{N}{2}\int \omega_k \frac{dk}{2\pi}\right)\left[1 + \frac{N\lambda}{4}\int \frac{dk}{2\omega_k 2\pi}\right]^2 + \frac{N}{2}\int \frac{k^2 dk}{\omega_k 2\pi}\left[1 + \frac{N\lambda}{4}\int \frac{dk}{2\omega_k 2\pi}\right]^{-2} \qquad (3.1)$$

minimizing functionally with respect to $\omega_\ell$ i.e. $\delta E_{vac}/\delta\omega_\ell = 0$ implies that the following integral equation for $\omega_\ell$ holds

$$0 = \frac{N}{2}\left[1 + \frac{\lambda N}{4}\int \frac{dk}{4\pi\omega_k}\right]^2$$

$$+ N\int \frac{\omega_k dk}{2\pi}\left(1 + \frac{\lambda N}{4}\int \frac{dk}{4\pi\omega_k}\right)\left(\frac{-N\lambda}{8\omega_\ell^2}\right)$$

$$- \frac{N}{2}\frac{\ell^2}{\omega_\ell^2}\left(1 + \frac{N\lambda}{4}\int \frac{dk}{4\pi\omega_k}\right)^{-2}$$

$$+ \frac{N^2\lambda}{8\omega_\ell^2}\int \frac{k^2 dk}{4\pi\omega_k}\left(1 + \frac{N\lambda}{4}\int \frac{dk}{4\pi\omega_k}\right)^{-3}$$

We note that

$$< 0|\bar{\phi}\phi|0 > = N\int \frac{dk}{4\pi\omega_k}$$

and try a solution of the form $k^2 = A\omega_k^2 - m^2$. The conditions for a solution are

$$A = (1 + \frac{\lambda}{4} < \bar{\phi}\phi >)^4 \equiv <D>^4$$

and

$$\frac{m^2}{\omega_\ell^2 I} \left(1 - \frac{\lambda < \bar{\phi}\phi >}{2<D>}\right) = 0$$

Therefore either $m = 0$ or $< \bar{\phi}\phi > = 4/\lambda$ corresponding to the two possible solutions in the Lagrangian case. By Coleman's theorem[11], we know that in 2 dimensions the vacuum will always pick the second solution. Note that in the case of $<\bar{\phi}\phi> = 4/\lambda$, we may compute the above integrals using dimensional regularization and solve for the mass m. The result is the same as obtained previously (see eq. 2.3).



In the $CP^{N-1}$ case there are two additional terms in the Hamiltonian, but as previously discussed they make no contribution to the vacuum energy in the large N limit. Consequently we have the same structure for the vacuum, as in the Lagrangian approach. However it is important to notice that the vacuum has chosen to sit on the apparent singularity in the Hamiltonian. There are two attitudes that one could adopt at this point. The Hamiltonian may not be valid for these particular field configurations since it is derived from a Lagrangian which contains some redundancy in the form of a residual gauge invariance. Consequently a new gauge fixing should be adopted to eliminate this gauge freedom. This will be discussed later. The alternative (and more appealing) attitude is to say that this singularity is a signal for confinement, as we find that charged one particle states have infinite energy in this gauge and that thus do not constitute the lowest excitations to the vacuum. We are not able to explicitly construct the lowest energy excitations of the vacuum but we can outline the structure of a finite energy state. This finite energy state is a particle - anti-particle bound state which resembles a quantum bag. This state is in the naive 1/N vacuum outside the line-section (-a,a) (its charge density outside this sector is identically zero, voiding the effect of the pole). All its energy comes from charge fluctuations in the (-a, a) region. The infinite energy is avoided by being in the asymptotically free perturbation theory vacuum $< \bar{\phi}\phi >= 0$ in this region. The excitation energies of this bag will essentially follow the bound state energies of a linear Coulomb potential. Note that similar to the Lagrangian case we are assuming that $< \frac{1}{\bar{\phi}\phi} > = \frac{1}{<\bar{\phi}\phi>} + 0\,(\frac{1}{N})$ .

We conclude this section by mentioning an attempt to construct the Hamiltonian in an axial gauge. The axial gauge condition $A_1 = 0$ can be implemented by solving for $z_N$ in

$$\bar{z}_N \overleftrightarrow{\partial}_1 z_N + \bar{z}_i \overleftrightarrow{\partial}_1 z_i = 0$$

with $i = 1, \ldots N - 1$

i.e.

$$z_N(x) = \left( \frac{4}{\lambda} - \bar{z}_i(x) z_i(x) \right)^{\frac{1}{2}} \exp - \frac{i}{2} \int_{x_0}^{x} dy \left[ \frac{i\bar{z}_i(y) \overleftrightarrow{\partial}_y z_i(y)}{\frac{4}{\lambda} - \bar{z}_i(y) z_i(y)} \right]$$

We did not succeed in expressing the Hamiltonian in terms of unconstrained variables, nevertheless we suspect that the symmetry is restored by having the expectation value of $z_N$ vanish. For the naive $1/N$ limit it vanishes because $<\bar{z}_i z_i>$ becomes of order $4/\lambda$ while to next order $z_N$ vanishes because the phase factor wildly oscillates. These large oscillations are related to the formation of a linear potential emphasizing terms of the form $\bar{\phi}\overleftrightarrow{\partial}_1\phi$. Our failure to write down the Hamiltonian is unfortunate because we expect to obtain a long range force directly in axial gauge.

## 4. SPECTRUM IN THE 1/N LIMIT

In this section we concentrate on two points. We first calulate the splitting between the $N^2$-1 particles in the adjoint representation of SU(N) and those in the singlet representation. The $N^2$ bound states split such that the adjoint representation has lower energy.



We then point out the non smooth connection between the large N and large coupling limits of $CP^{N-1}$.

There are two contributions to the potential between a $z_i$ particle and an antiparticle, a long range piece and a short range piece. In the limit of large N and large mass the system is non-relativistic and may be solved by a Schroedinger equation. We will perform the calculation in the language of Ref. 11. The interaction between $\bar{z}_i$ and $z_j$ when $i \neq j$ is due to t channel exchange of the $A_\mu$ field. This gives rise to a potential at large separation

$$v(r) = \frac{12\pi m^2}{N} |r|$$

where m is the mass of the $z_i$ quanta. The energy levels are given by the solution of the Schroedinger equation

$$-\frac{1}{m} \frac{d^2\psi(x)}{dx^2} + \left(\frac{12\pi m^2}{N}|x| - E\right)\psi(x) = 0$$

where E is the binding energy of the state. The unnormalized wave functions are

$$\psi(x) = \begin{cases} A(g^{1/3}x - g^{-2/3}mE), & x \geq 0 \\ A(-g^{1/3}x - g^{-2/3}mE), & x \leq 0 \end{cases}$$

with

$$g = \frac{12\pi m^3}{N}$$

and A(x) is an Airy function. The eigenvalue conditions are:

even parity: $A'(-g^{-2/3}mE) = 0$

odd parity: $A(-g^{-2/3}mE) = 0$

where a prime denotes differentation with respect to the arguments. The ground state is of even parity; thereafter the states are of alternating parity.

Now in the case of $i = j$ there is an additional interaction contribution to the potential. This is a contact term due to the s channel propogation of $A_\mu$ and the short range field $\alpha$. In fact when $s = 4m^2$ the scalar field interaction is zero,* and the additional contribution to the potential is

$$v(r) = - m^2 \delta(r)$$

This potential is repulsive and naively of order one in the 1/N expansion; dimensional analysis corrects this false impression.[12] Thus the eigenvalue condition for the singlet states is

even parity $2g^{1/3}A'(-g^{-2/3}mE) = m^3 A(-g^{-2/3}mE)$

odd parity $A(-g^{-2/3}mE) = 0$

The lowest singlet is still even parity but it is higher than the corresponding adjoint state for which there is no contact term. The spectrum in the large N limit is indicated in Figure 7. Several comments should be made about this spectrum. The ground state is an adjoint and there is no degeneracy in the leading order.

* This can be seen immediately from the four-point function given in Section II.



On the other hand the second even parity state is degenerate with the first odd parity state in the large N limit. It is conceivable that the N = 2 case (equivalent to the 0(3)/0(2) non-linear sigma model) is reached smoothly by the higher states becoming unstable as N is reduced. Secondly we can estimate the value of N for which the non-relativistic calculation is valid:

$$N \gg 12\ (1.06)^{3/2} \approx 40$$

Finally we contrast this spectrum with that obtained from the strong coupling limit of the lattice version of the theory as formulated by Stone.[8] His spectrum is generated by single site excitations is indicated in Figure 8a. The energies are proportional to the eigenvalues of the quadratic Casimir operators in the various representations. We note however that the energy of two adjoint representations on different sites (E = 2N) is less than that of the first excited state given by Stone (2N + 2), so the spectrum should really be that in Figure 8b. The ground state is clearly the same as that obtained in the large N limit. It is also clear that for the higher lying states the limits of large N and strong coupling on the lattice do not commute. There must be considerable level crossing if the states are to correspond.

## CONCLUSIONS

We have looked at the $CP^{N-1}$ model in terms of a non-linear representation having unconstrained variables. The Lagrangian was treated in a unitary gauge and a gauge ambiguity emerged in the form of a residual gauge invariance for a certain class of field configurations. This ambiguity appears to be similar to the Gribov ambiguity in QCD. A solution of the model in the $1/N$ limit shows that the vacuum chooses field configurations which possess the ambiguity (and which give $CP^{N-1}$ its non-trivial topological structure). This is reflected in the appearence of a gauge field with dynamics. The ambiguity appears in a much more dramatic manner in the Hamiltonian formulation. A pole appears at value of the field having the ambiguity.

We are able to determine the $1/N$ vacuum of the Hamiltonian by means of a Hartree approximation (useful for the $O(N)/O(N-1)$ case also). The pole appears to be a signal for confinement in that its effects can be eliminated by considering only charge neutral states. We show that bound states with bag like properties have finite energy, although attempts to obtain the long range-force in an axial gauge should still be made. The spectrum calculated in the non-relativistic limit (valid for $N \gg 40$) reveals that the ground state is in the adjoint representation of $SU(N)$. The comparison with the lattice version of the model in the strong coupling limit shows that although the ground states are the same, the higher lying states are radically different and the connection between the two limits is not smooth. The next step would be to complete the linearization of the system, namely to obtain $CP^{N-1}$ for large $N$ as a scalar field theory of charge neutral states.

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## FIGURE CAPTIONS

Figure 1: To leading order in $\frac{1}{N}$, $:(\bar{\phi}\phi)^n: = (:\bar{\phi}\phi:)^n + O(\frac{1}{N})$ where $\bar{\phi}\phi \equiv \sum_{i=1}^{N} \phi_i\phi_i$. This can be illustrated schematically by the fact that graph (a) is down by a factor $\frac{1}{N}$ relative to graph (b), where the color label flows as indicated by the arrow.

Figure 2: The bare four-point vertex $i\Gamma^{(4)}_{ijk\ell}$. We decompose $\Gamma^{(4)}_{ijk\ell} = \Gamma^{(4)}_s \delta_{ij}\delta_{k\ell} + \Gamma^{(4)}_t \delta_{ik}\delta_{j\ell} + \Gamma^{(4)}_u \delta_{i\ell}\delta_{jk}$ and analyze the separate pieces in the text.

Figure 3: Diagramatic illustration of the four types of terms given in Fig. 2.

Figure 4: Schematic representation of the $T_{ij}$ functions. The notation of Fig. 3 is used here. Note that for $i \neq j$, the $T_{ji}$ are simply reflections of $T_{ij}$ and hence are not displayed above.

Figure 5: Graphical representation of the matrix $\Sigma$.

Figure 6: Graphical representation of the term which leads to a long range force.

Figure 7: Spectrum of bound states in the large $N$ limit. The SU(N) representation and parity of the lowest lying states are schematically indicated. The energy increases from bottom to top (scale is arbitrary).

Figure 8: Spectrum of bound states in the strong coupling limit of the lattice version of the theory (see Ref. 8).

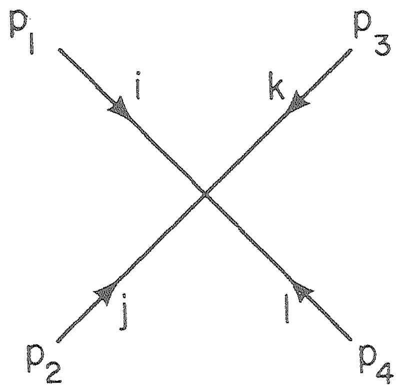
(a) Spectrum due to single-site excitation alone.

(b) Spectrum due to excitation of one and more than one site.



The SU(N) representations of the levels are given with the following notation: Corresponding to a Young diagram consisting of $n_k$ boxes in the $k^{th}$ row $(n_1 \geq n_2 \geq \cdots \geq n_N)$ we have the representation $[p_1, p_2, \cdots, p_{N-1}]$ where $p_k = n_{k+1} - n_k$. In this notation, the SU(N) adjoint is $[1,0,\cdots,0,1]$, i.e. $p_1 = p_{N-1} = 1$, $p_k = 0$ for all other $k$. Finally, by $\mathrm{SYM}(A \otimes B)$, we mean the SU(N) representations contained in the symmetric part of the tensor product of representations A and B.


Figure 1




$$i\lambda\left[\left(p_1\cdot p_2+p_3\cdot p_4+\frac{3m^2}{2}+(p_1+p_2)\cdot(p_3+p_4)\right)\delta_{ij}\,\delta_{kl}+\text{permutations}\right]$$

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Figure 2

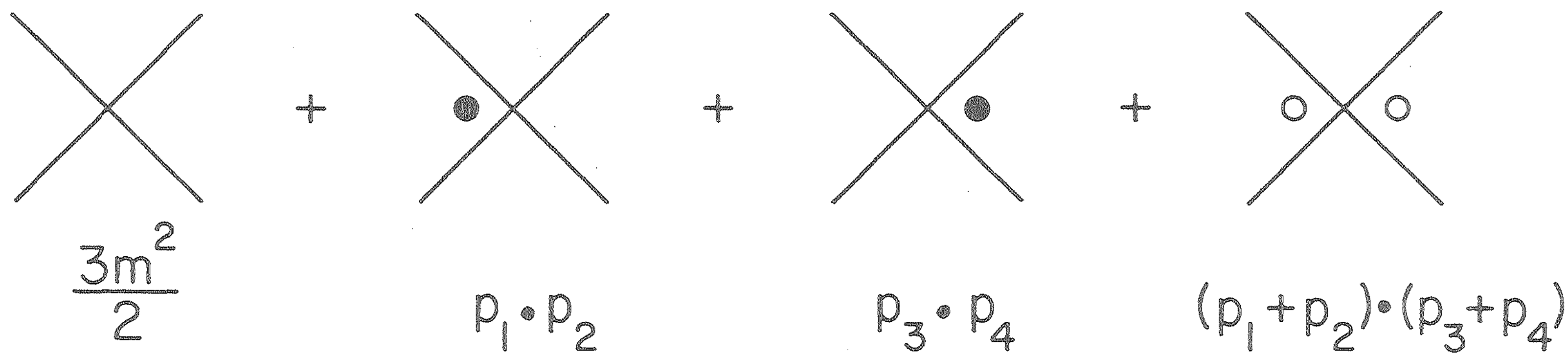


Figure 3

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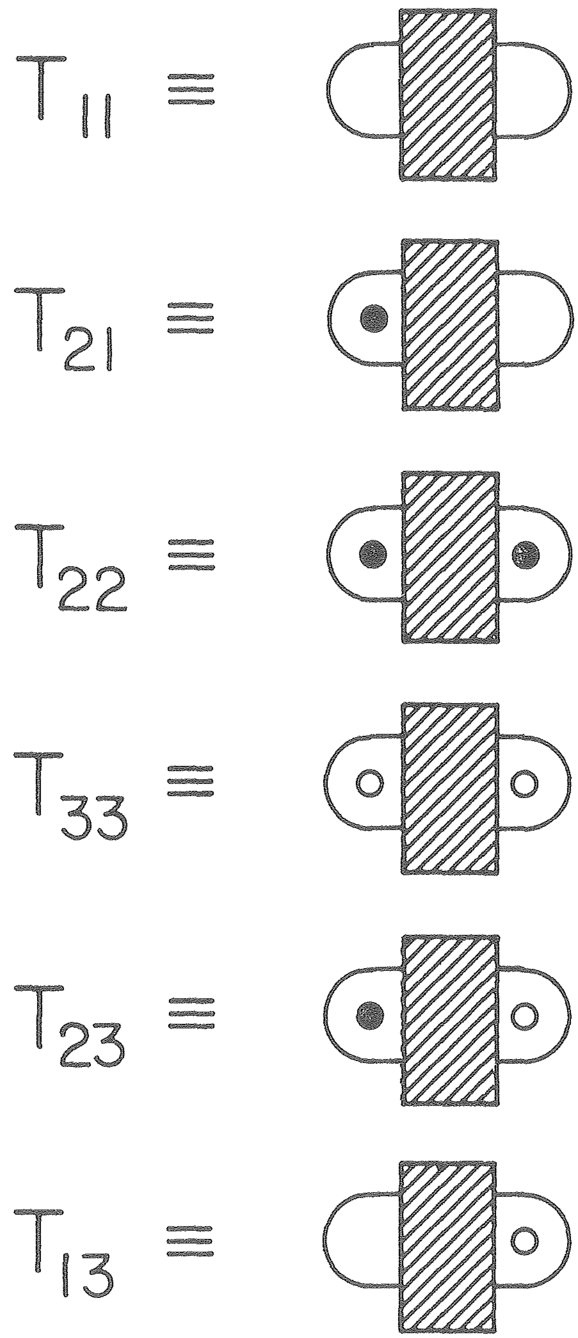


XBL 802-357

Figure 4

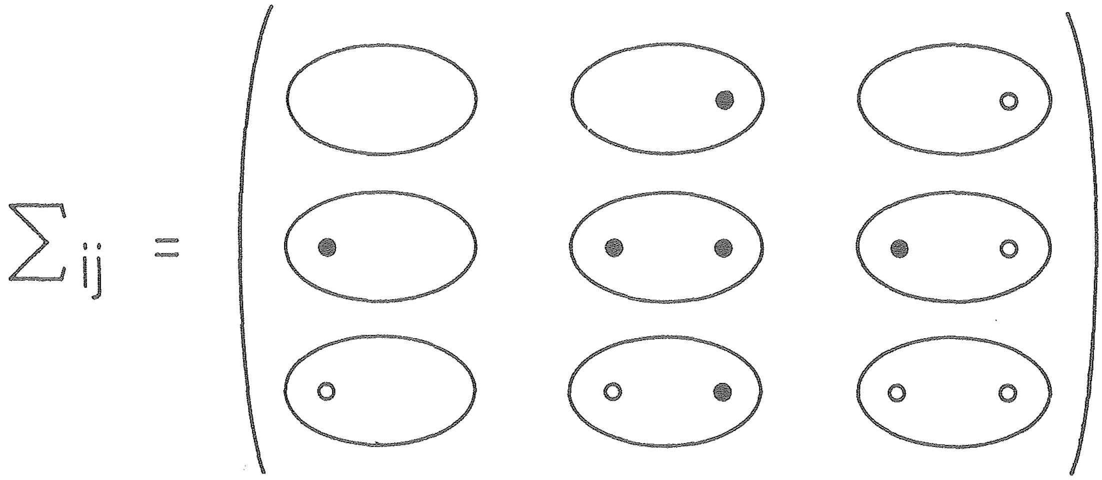


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Figure 5

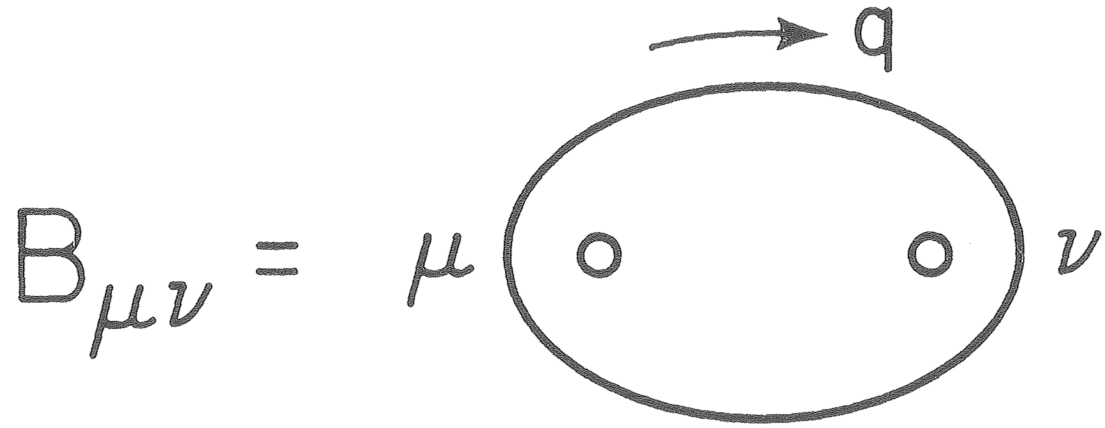


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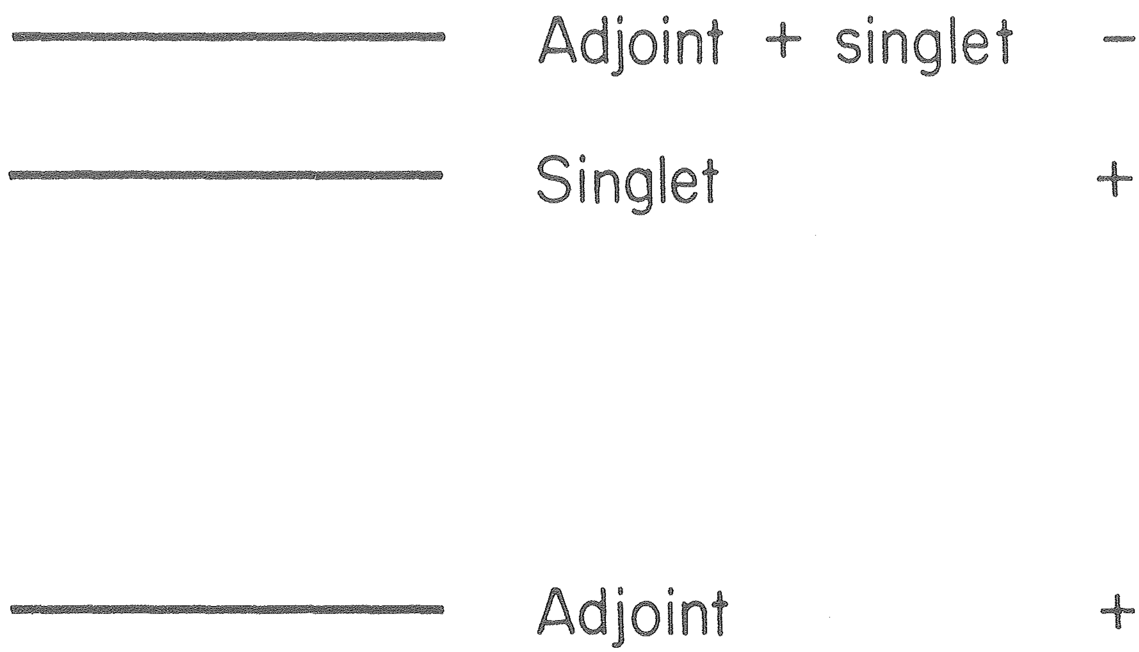
Figure 6




XBL 802-363

Figure 7

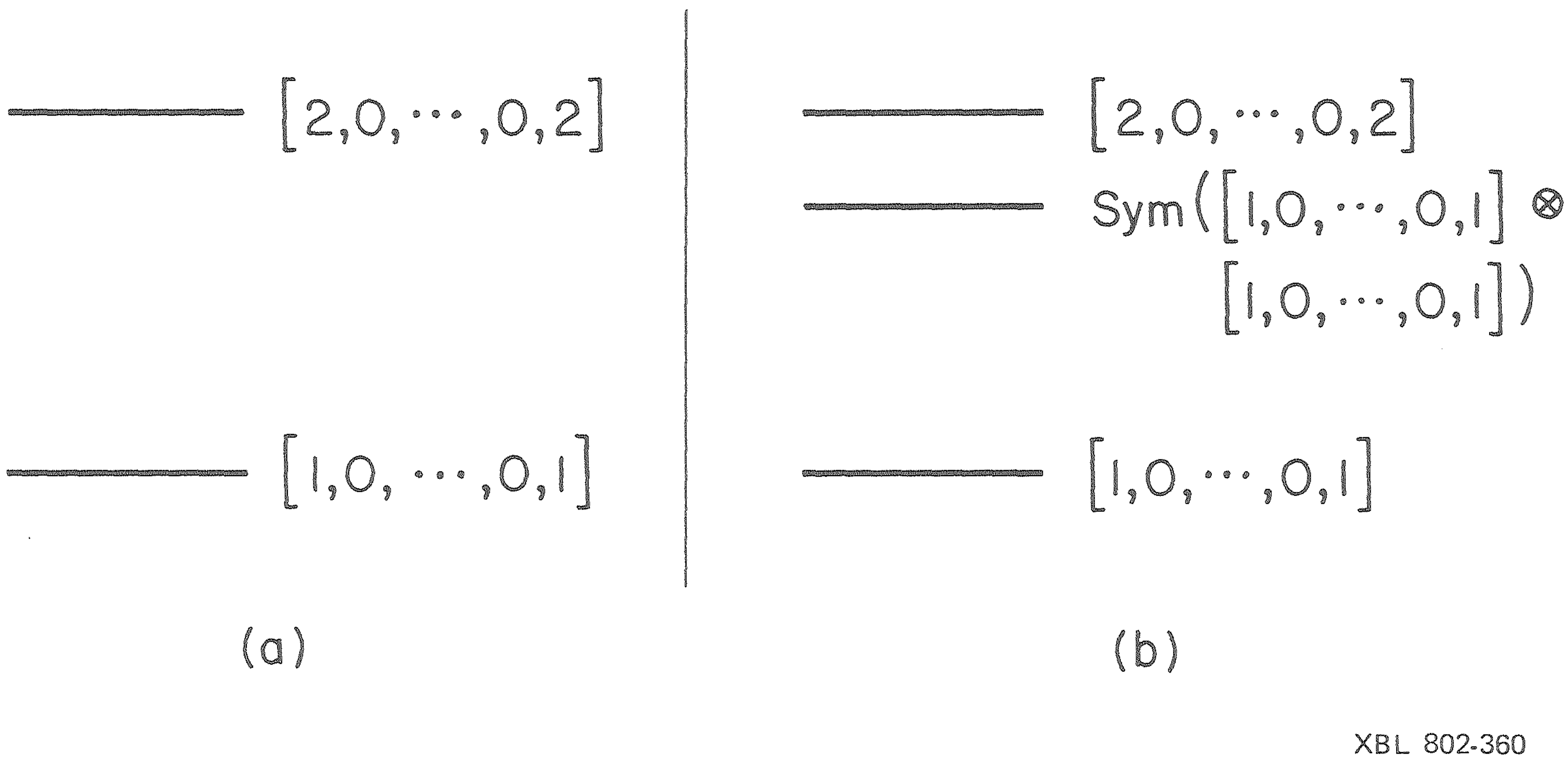



Figure 8